# Learning with norm-based neural networks: model capacity, function spaces, and computational-statistical gaps


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[joint work with Leello Dadi, Zhenyu Zhu, Volkan Cevher (EPFL)]


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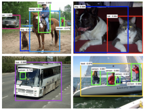
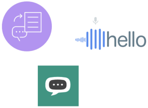
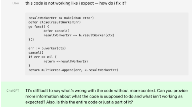
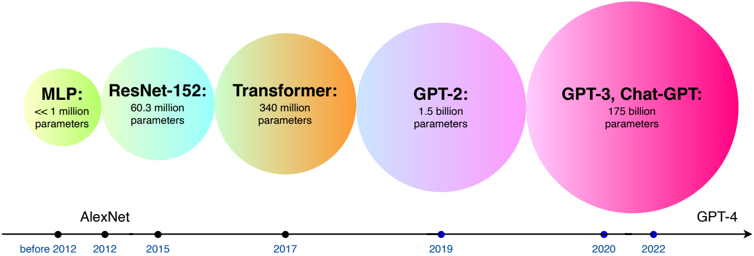
# In the era of deep learning





# Scaling law: under compute budget

**scaling law [13]**

$$\text{test loss} = A \quad \text{Model Size}^{a} + B \quad \text{Data Size}^{b} + C$$

under limited compute budget

- data-parameter trade-off
- time-space trade-off

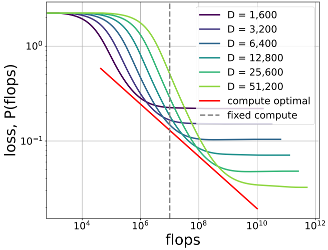
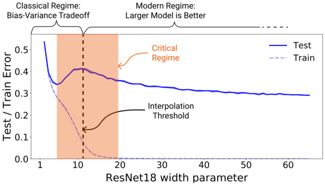


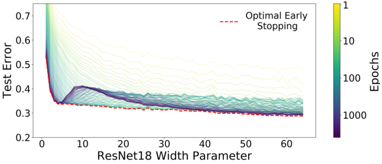

Figure 1: Scaling law under compute-optimal configuration [21].

# Model size is a "right" complexity?

- double descent [4] (Belkin, Hsu, Ma, Mandal, 2019)




(a) Results on ResNet18 [18]




(b) Optimal early stopping [18].

- Empirically: neural network pruning [16], lottery ticket hypothesis [11], fine-tuning with large dropout [28]
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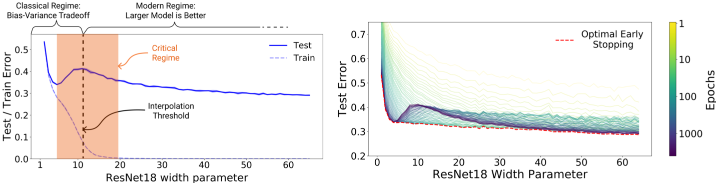


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# What is the "right" model complexity?

Complexity of a prediction rule, e.g.,

- number of parameters
- norm of parameters

[2] (Bartlett, 1998)

The size of the weights is more important than the size of the network!

Norm-based capacity:[19, 24, 20, 8]

| name | definition | rank correlation |
|---|---|---|
| Parameter Frobenius norm | $\sum_{i=1}^{L} \|W_i\|_F^2$ | 0.073 |
| Frobenius distance to initialization [17] | $\sum_{i=1}^{L} \|W_i - W_i^0\|_F^2$ | 0.263 |
| Spectral complexity [3] | $\prod_{i=1}^{L} \|W_i\| \left( \sum_{i=1}^{L} \frac{\|W_i\|_{2,1}^{3/2}}{\|W_i\|^{3/2}} \right)^{2/3}$ | 0.537 |
| Fisher-Rao [14] | $\frac{(L+1)^2}{n} \sum_{i=1}^{n} \langle W, \nabla_W \ell(h_W(x_i), y_i) \rangle$ | 0.078 |
| Path-norm [19] | $\sum_{(i_0,\dots,i_L)} \left( \prod_{j=1}^{L} W_{i_j, i_{j-1}} \right)^2$ | 0.373 |

Table 1: Complexity measures compared in the empirical study [12], and their correlation with generalization

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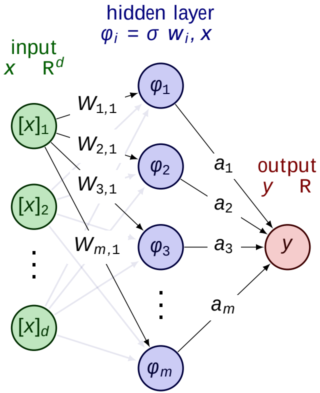


$\mathcal{P}_m = \{ f(x) := \frac{1}{m} \sum_{k=1}^{m} a_k \, \sigma(\langle w_k, x\rangle) \}$

**$\ell_1$-path norm**

$\|f\|_P := \frac{1}{m} \sum_{k=1}^{m} |a_k| \|w_k\|_1$

- semi-norm
- representation cost
- relations to Barron spaces $\mathcal{B}$ [1, 10]
- $\|f\|_{\mathcal{B}} \le \|f\|_P \le 2\|f\|_{\mathcal{B}}$

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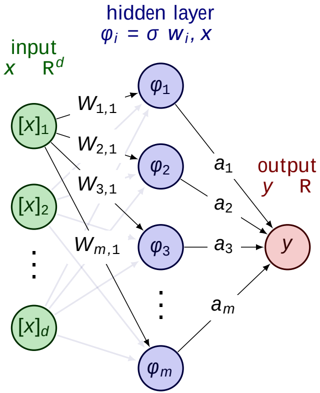


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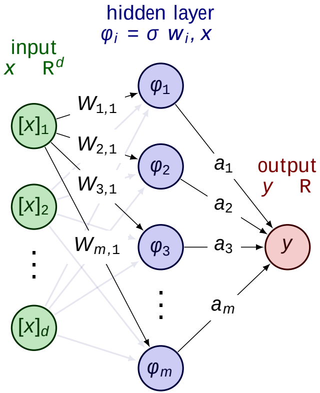


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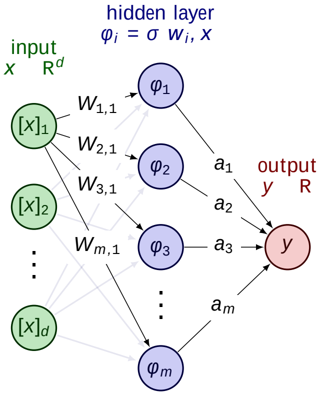


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Consider a random features model [22, 15]

- first layer: $w \overset{iid}{\sim} \pi \in P(\mathcal{W})$; only train the second layer

infinite many features $f_a(x) = \int_{\mathcal{W}} a(w)\phi(x; w) \mathrm{d}\pi(w)$

**Definition (RKHS and Barron space [9, 5])**

$$\mathcal{F}_{p,\pi} := \{f_a : \|a\|_{L^p(\pi)} < \infty\}, \quad \|f\|_{\mathcal{F}_{p,\pi}} := \inf_{f = f_a} \|a\|_{L^p(\pi)}$$

For any $1 \le p \le \infty$, we have

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# Our results: statistical guarantees

For the class of two-layer neural networks $\mathcal{G}_R = \{f \in \mathcal{P}_m : \|f\|_{\mathcal{P}} \le R\}$

$$\hat{f} := \operatorname*{argmin}_{f \in \mathcal{G}_R} \frac{1}{n} \sum_{i=1}^{n} (y_i - f(x_i))^2 .$$

**Theorem (Liu, Dadi, Cevher, JMLR 2024)**

*Under standard assumptions (bounded data, $f^\star \in \mathcal{B}$), for two-layer over-parameterized neural networks, we have*

$$\|\hat{f} - f^\star\|^2_{L^2_{\rho_X}} \lesssim \frac{R^2}{m} + R^2 d^{\frac{1}{3}} n^{-\frac{d+2}{2d+2}} \quad w.h.p.$$

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**Proposition (metric entropy)**

*For bounded data* $\|x\|_\infty \leq 1$, *denote* $G_R = \{f \in P_m : \|f\|_P \leqslant R\}$, *the metric entropy of* $G_1$ *can be bounded by*

$$\log N_2(G_1, \epsilon) \leqslant C d\, \epsilon^{-\frac{2d}{d+2}}, \quad \forall \epsilon > 0 \quad \text{and} \quad d \geq 5,$$

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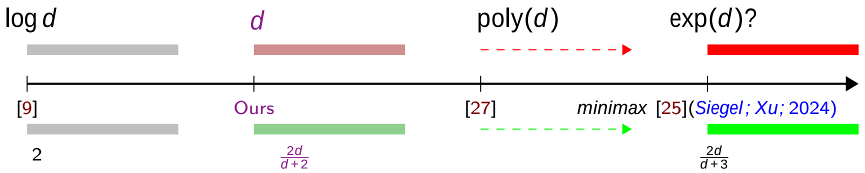

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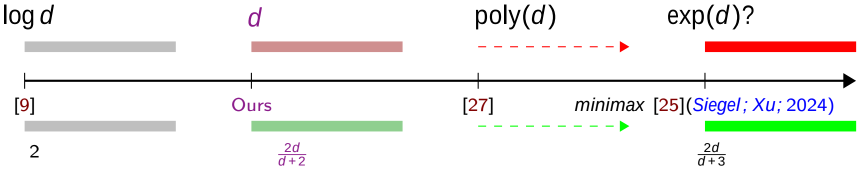


The "best" trade-off between $\epsilon$ and $d$.

# Computational-to-statistical gaps

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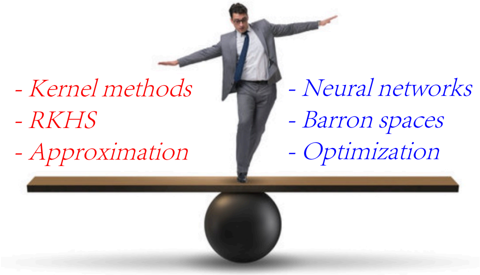

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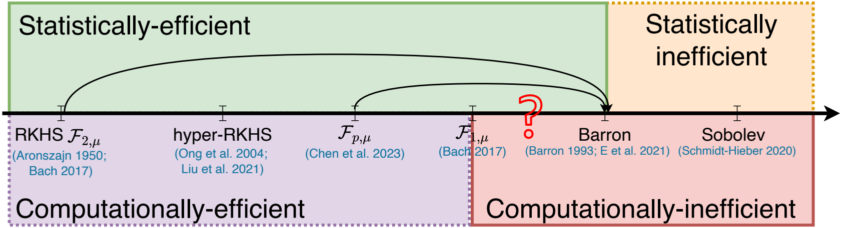

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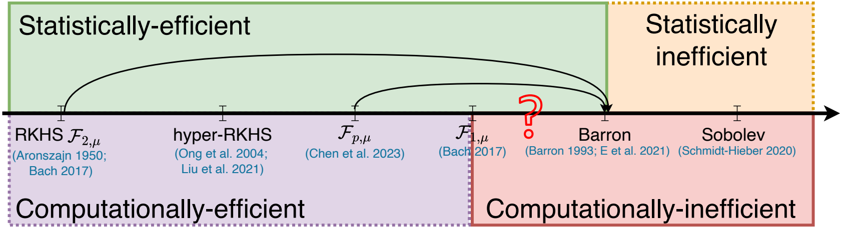


Do some Barron functions can be learned by two-layer NNs, both statistically and computationally e□cient?

# Learning with multiple ReLU neurons

Can we learn multiple ReLU neurons by two-layer NNs, both statistically and computationally efficient?

$$f^\star(x) = \sum_{j=1}^{k} a_j \sigma(\langle v_j, x \rangle), k = O(1)$$

$\|\hat{f} - f^\star\|_{L^2(\gamma_d)} \leq \epsilon$ from $\{x_i, f^\star(x_i)\}_{i=1}^n$ with $x_i \sim \mathcal{N}(0, I_d)$

**Theorem ([6] PAC learning $f^\star$ under Gaussian measure)**

*There exists an algorithm that requires time/samples at $(d/\epsilon)^{O(k^2)}$*

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**Theorem ([6] PAC learning $f^{\star}$ under Gaussian measure)**

*There exists an algorithm that requires time/samples at $(d/\ )^{O(k^2)}$*

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## Learning with multiple ReLU neurons

Can we learn multiple ReLU neurons by two-layer NNs, both statistically and computationally efficient?

$$f^\star(x) = \sum_{j=1}^{k} a_j \sigma(\langle v_j, x \rangle), \quad k = O(1)$$

$\|\hat{f} - f^\star\|_{L^2(\mu_d)} \leq \epsilon$ from $\{x_i, f^\star(x_i)\}_{i=1}^n$ with $x_i \sim \mathcal{N}(0, I_d)$

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# How does student(s) become teacher(s) under GD training?

**Learning multi ReLU neurons by two-layer NN via online SGD**

$$L(W) = \frac{1}{2}\mathbb{E}_{x \sim N(0, I_d)} \left( \sum_{i=1}^{m} \sigma(\langle w_i, x \rangle) - f^{\star}(x) \right)^2$$

- Gaussian initialization $w_i \sim N(0, \sigma^2 I_d)$
- angle: $\theta_{ij} := \angle(w_i, v_j)$

**Assumption**

- *diverse teacher neurons: $\{v_j\}_{j=1}^{k}$ are orthogonal and $\|v_j\|_2 = \mathtt{const}$*
- *warm start: the smallest angle not close to orthogonal*
  $\Rightarrow$ *weak recovery: $\langle w_i, v_{i^\star} \rangle \gg \langle w_i, v_j \rangle$*

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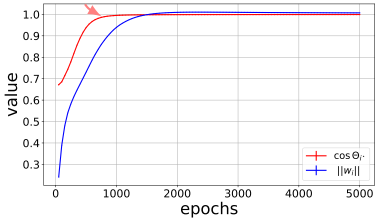


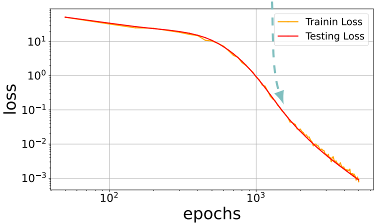


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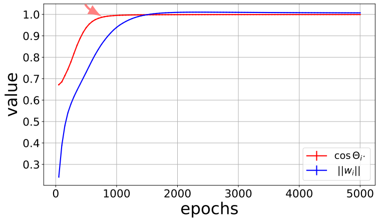


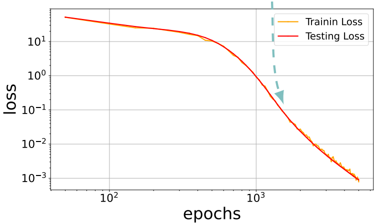


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# Take-away messages

- model size -> size of weights -> path norm -> Barron spaces
- statistical guarantees with improved sample complexity
- computational-statistical gap -> learning with multiple ReLU neurons

**We're organizing one workshop at NeurIPS 2024!**

Fine-Tuning in Modern Machine Learning: Principles and Scalability

https://sites.google.com/view/neurips2024-ftw/home

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