Discrete Mathematics and Its Applications 2 (CS147)

Lecture 4: Bubble-sort

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Logistics: Homework 1...

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Lectures and Seminars

Relevant Textbooks and Lecture Notes:

- [DPV] Algorithms.
- [Guichard] An Introduction to Combinatorics and Graph Theory.
- [Aspnes] Notes on Discrete Mathematics.
- [Mitzenmacher and Upfal] Probability and Computing.
- [Durrett] Elementary Probability for Applications.
- [GW] Probability: An Introduction.

Assessment:

Final exam (80%), Two Homework Assignments (10% each).

- Assignment 1: [homework1.pdf] **deadline:** Mon 5 Feb, 2024 (12pm GMT)

LaTeX template for answering questions if you prefer [template.tex]

- Assignment 2:

Contents of Lectures and Seminars:

- Available at this page.

Try to answer ALL questions. Before you start, carefully read the following instructions below:

- If you are writing the answers by hand, then you must:
  - (1) Write them on A4 pages.
  - (2) Convert them into .pdf formats (for example, by taking pictures from your phones and then using some freely available online tool which converts .jpeg files into .pdf files).
  - (3) Merge all of them into a **SINGLE .pdf** file.
  - (4) Ensure that your handwriting (as captured in the .pdf file) is clear and legible.

  Otherwise, if you want, then you are also free to type the solutions in a computer and then convert them into a SINGLE .pdf file (using some free online tool for converting files into .pdf formats).

  Remark: If you would like to type the solutions in a computer via LaTeX, I suggest to use the online LaTeX editor Overleaf, see a tutorial on `https://www.overleaf.com/learn/latex/Tutorials`. A template for Assignment 1 can be found on the module webpage with basic common and useful latex constructs. When finishing the homework, you need to download this .pdf file and upload it on tabula.

- Answer each question in a separate page, and complete the answer of a given question within a single page. Thus, overall the file you upload in tabula should consist of **(at most) 5 pages**, where each page should be devoted to the answer of one question. If you take more than one page to answer a question, then only the first page of the answer will be marked.
- Write down your **university ID number** on the top of each page.
- Upload your answers in tabula in a SINGLE file in .pdf format.

If you fail to follow any of these instructions, then your submission may not be marked at all.

Further comments:

- As a general rule, try to be precise and to the point in your answers, instead of being verbose and writing long paragraphs to explain something that can be concisely summarised in a few sentences.

Clarifications in Lecture 2: Example

Example

Given $f(n) = 2^{\sqrt{\ln n}}$ and $g(n) = n^{0.0001}$, check $f(n) = \mathcal{O}(g(n))$ and $f(n) \neq \Omega(g(n))$.

Clarifications in Lecture 2: Example

Example

Given $f(n) = 2^{\sqrt{\ln n}}$ and $g(n) = n^{0.0001}$, check $f(n) = \mathcal{O}(g(n))$ and $f(n) \neq \Omega(g(n))$.

Proof.

Take the logarithmic operation of $f(n)$ and $g(n)$...
Updated: We can prove it by the definition. We prove $f(n) = \mathcal{O}(g(n))$ as an example, that means, we need to find $c > 0$ and $n > N$ such that $f(n) \leq cg(n)$

$$2^{\sqrt{\ln n}} \leq cn^{0.0001} \quad \Leftrightarrow \sqrt{\ln n}\ln 2 \leq \ln c + 0.0001 \ln n \quad \text{[taking } c \geq 1 \text{ for nonnegativity]}$$

$$\Leftrightarrow \sqrt{\ln n}\ln 2 \leq 0.0001 \ln n \quad \text{[taking } c = 1 \text{ for simplicity]}$$

$$\Leftrightarrow \sqrt{\ln n} \geq 10^4 \ln 2 \quad \Leftrightarrow n \geq e^{10^8 (\ln 2)^2} := N\,.$$

We conclude the proof by taking $c = 1$ and $N := e^{10^8(\ln 2)^2}$. □

Sorting (Problem definition)

- Input: Array $A[1, 2, \ldots, n]$ of n numbers.
- Output: The same array with the same numbers, but sorted in an increasing order.

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Example

Input: A { [7] [6] [8] [4] [9] }
Output: A { [4] [6] [7] [8] [9] }

Sorting (Problem definition)

- Input: Array $A[1, 2, \ldots, n]$ of n numbers.
- Output: The same array with the same numbers, but sorted in an increasing order.

Example

Input: A { [7] [6] [8] [4] [9] }
Output: A { [4] [6] [7] [8] [9] }

Assumption

We will assume that $A[k] \neq A[l]$ for all $k, l \in [1, n]$, $k \neq l$.

It is straightforward to extend our analysis when the input array contains duplicate entries.

Main idea behind Bubble-sort

- Keep comparing successive entries $A[j]$ and $A[j+1]$.

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The entries are in correct sorted order.

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Key message

Bubble: We always put the larger number to the end.

Intuitive understanding (first round)

[3] [1] [5] [6] [7] [2] [8] [4]

first round:

[3] [1] [5] [6] [7] [2] [8] [4]

Intuitive understanding (first round)

[3] [1] [5] [6] [7] [2] [8] [4]

first round:

[3] [1] [5] [6] [7] [2] [8] [4]

[1] [3] [5] [6] [7] [2] [4] [8]

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[3] [1] [5] [6] [7] [2] [8] [4]

first round:

[3] [1] [5] [6] [7] [2] [8] [4]

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Intuitive understanding (first round)

[3] [1] [5] [6] [7] [2] [8] [4]

first round:

[3] [1] [5] [6] [7] [2] [8] [4]

[1] [3] [5] [6] [7] [2] [4] [8]

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Intuitive understanding (first round)

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[1] [3] [5] [6] [7] [2] [4] [8]

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[1] [3] [5] [6] [7] [2] [4] [8]

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[1] [3] [5] [6] [7] [2] [8] [4]

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Intuitive understanding (first round)

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[1] [3] [5] [6] [2] [7] [8] [4]

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Statement (Observation)

do 7 times comparison and put the **largest** *number to the end!*

Intuition understanding (second round)

[1] [3] [5] [6] [2] [7] [4] [8]

second round:

[1] [3] [5] [6] [7] [2] [4] [8]

Intuition understanding (second round)

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second round:

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second round:

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Intuition understanding (second round)

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second round:

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Intuition understanding (second round)

[1] [3] [5] [6] [2] [7] [4] [8]

second round:

[1] [3] [5] [6] [7] [2] [4] [8]

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[1] [3] [5] [6] [7] [2] [4] [8]

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[1] [3] [5] [6] [2] [7] [4] [8]

[1] [3] [5] [6] [2] [4] [7] [8]

Statement (Observation)

For these 7 numbers: do 6 times comparison and put the **largest** *number to the end!*
$\Rightarrow$ *The last two numbers are sorted!*

Intuition understanding (third round)

[1] [3] [5] [6] [2] [4] [7] [8]

third round:

[1] [3] [5] [6] [2] [4] [7] [8]

Intuition understanding (third round)

[1] [3] [5] [6] [2] [4] [7] [8]

third round:

[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [6] [2] [4] [7] [8]

Intuition understanding (third round)

[1] [3] [5] [6] [2] [4] [7] [8]

third round:

[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [6] [2] [4] [7] [8]

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Intuition understanding (third round)

[1] [3] [5] [6] [2] [4] [7] [8]

third round:

[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [6] [2] [4] [7] [8]

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[1] [3] [5] [6] [2] [4] [7] [8]

Intuition understanding (third round)

[1] [3] [5] [6] [2] [4] [7] [8]

third round:

[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [6] [2] [4] [7] [8]

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[1] [3] [5] [2] [6] [4] [7] [8]

Intuition understanding (third round)

[1] [3] [5] [6] [2] [4] [7] [8]

third round:

[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [6] [2] [4] [7] [8]

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[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [2] [6] [4] [7] [8]

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Intuition understanding (third round)

[1] [3] [5] [6] [2] [4] [7] [8]

third round:

[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [6] [2] [4] [7] [8]

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[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [2] [6] [4] [7] [8]

[1] [3] [5] [2] [6] [4] [7] [8]

[1] [3] [5] [2] [4] [6] [7] [8]

Intuition understanding (third round)

[1] [3] [5] [6] [2] [4] [7] [8]

third round:

[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [6] [2] [4] [7] [8]

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[1] [3] [5] [6] [2] [4] [7] [8]

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[1] [3] [5] [2] [6] [4] [7] [8]

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Intuition understanding (third round)

[1] [3] [5] [6] [2] [4] [7] [8]

third round:

[1] [3] [5] [6] [2] [4] [7] [8]

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[1] [3] [5] [6] [2] [4] [7] [8]

[1] [3] [5] [2] [6] [4] [7] [8]

[1] [3] [5] [2] [6] [4] [7] [8]

[1] [3] [5] [2] [4] [6] [7] [8]

[1] [3] [5] [2] [4] [6] [7] [8]

Statement (Observation)

For these 6 numbers: do 5 times comparison and put the **largest** *number to the end!*

$\Rightarrow$ *The last 3 numbers are sorted!*

Intuition understanding (the 4th round)

[1] [3] [5] [2] [4] [6] [7] [8]

4th round:

[1] [3] [5] [2] [4] [6] [7] [8]

Intuition understanding (the 4th round)

[1] [3] [5] [2] [4] [6] [7] [8]

4th round:

[1] [3] [5] [2] [4] [6] [7] [8]

[1] [3] [5] [2] [4] [6] [7] [8]

Intuition understanding (the 4th round)

[1] [3] [5] [2] [4] [6] [7] [8]

4th round:

[1] [3] [5] [2] [4] [6] [7] [8]

[1] [3] [5] [2] [4] [6] [7] [8]

...

Intuition understanding (the 4th round)

[1] [3] [5] [2] [4] [6] [7] [8]

4th round:

[1] [3] [5] [2] [4] [6] [7] [8]

[1] [3] [5] [2] [4] [6] [7] [8]

...

[1] [3] [5] [2] [4] [6] [7] [8]

Intuition understanding (the 4th round)

[1] [3] [5] [2] [4] [6] [7] [8]

4th round:

[1] [3] [5] [2] [4] [6] [7] [8]

[1] [3] [5] [2] [4] [6] [7] [8]

...

[1] [3] [5] [2] [4] [6] [7] [8]

[1] [3] [2] [5] [4] [6] [7] [8]

Intuition understanding (the 4th round)

[1] [3] [5] [2] [4] [6] [7] [8]

4th round:

[1] [3] [5] [2] [4] [6] [7] [8]

[1] [3] [5] [2] [4] [6] [7] [8]

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[1] [3] [5] [2] [4] [6] [7] [8]

[1] [3] [2] [5] [4] [6] [7] [8]

[1] [3] [2] [5] [4] [6] [7] [8]

Intuition understanding (the 4th round)

[1] [3] [5] [2] [4] [6] [7] [8]

4th round:

[1] [3] [5] [2] [4] [6] [7] [8]

[1] [3] [5] [2] [4] [6] [7] [8]

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[1] [3] [5] [2] [4] [6] [7] [8]

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4th round:

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[1] [3] [5] [2] [4] [6] [7] [8]

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Intuition understanding (the 4th round)

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4th round:

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[1] [3] [5] [2] [4] [6] [7] [8]

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[1] [3] [2] [4] [5] [6] [7] [8]

[1] [3] [2] [4] [5] [6] [7] [8]

Statement (Observation)

For these 5 numbers: do 4 times comparison and put the **largest** *number to the end!*
$\Rightarrow$ *The last 4 numbers are sorted!*

Intuition understanding (the 5th round)

[1] [3] [2] [4] [5] [6] [7] [8]

5th round:

[1] [3] [2] [4] [5] [6] [7] [8]

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[1] [2] [3] [4] [5] [6] [7] [8]

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Statement (Observation)

For these 4 numbers: do 3 times comparison and put the **largest** *number to the end!*
$\Rightarrow$ *The last 5 numbers are sorted!*

Intuition understanding (the 6th and 7th round)

[1] [2] [3] [4] [5] [6] [7] [8]

6th round:

[1] [2] [3] [4] [5] [6] [7] [8]

Intuition understanding (the 6th and 7th round)

[1] [2] [3] [4] [5] [6] [7] [8]

6th round:

[1] [2] [3] [4] [5] [6] [7] [8]

Statement (Observation)

For these 3 numbers: do 2 times comparison and put the **largest** *number to the end!*

$\Rightarrow$ *The last 6 numbers are sorted!*

Intuition understanding (the 6th and 7th round)

[1] [2] [3] [4] [5] [6] [7] [8]

6th round:

[1] [2] [3] [4] [5] [6] [7] [8]

Statement (Observation)

For these 3 numbers: do 2 times comparison and put the **largest** *number to the end!*

$\Rightarrow$ *The last 6 numbers are sorted!*

7th round:

[1] [2] [3] [4] [5] [6] [7] [8]

Intuition understanding (the 6th and 7th round)

[1] [2] [3] [4] [5] [6] [7] [8]

6th round:

[1] [2] [3] [4] [5] [6] [7] [8]

Statement (Observation)

For these 3 numbers: do 2 times comparison and put the **largest** *number to the end!*
$\Rightarrow$ *The last 6 numbers are sorted!*

7th round:

[1] [2] [3] [4] [5] [6] [7] [8]

Statement (Observation)

For these 2 numbers: do 1 times comparison and finish!

Bubble-sort (Pseudocode)

Algorithm 1: Bubble-sort

```
Input: An array A[1, 2, ..., n]
Output: An sorted array A[1, 2, ..., n]
for i = 1, ..., n − 1 do
    for j = 1, ..., n − i do
        if A[j] > A[j + 1] then
            X ← A[j];
            A[j] ← A[j + 1];
            A[j + 1] ← X;
        end
    end
end
```

Proof for correctness of Bubble-sort

- a) after one pass through the array, the largest entry will be at the end.
- b) if there is no pair of consecutive entries out of order, then the entire array is sorted.

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Proof for a) by induction.

1) After 1st comparison, $X := \max\{A[1], A[2]\}$, $A[2] \leftarrow X$.

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Proof for a) by induction.

1) After 1st comparison, $X := \max\{A[1], A[2]\}$, $A[2] \leftarrow X$.
2) After 2st comparison (and swap), $A[3]$ is the largest one among $A[1], A[2], A[3]$.

Proof for correctness of Bubble-sort

- a) after one pass through the array, the largest entry will be at the end.
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1) After 1st comparison, $X := \max\{A[1], A[2]\}$, $A[2] \leftarrow X$.
2) After 2st comparison (and swap), $A[3]$ is the largest one among $A[1], A[2], A[3]$.
3) Assume $A[k]$ is the largest one among $\{A[i]\}_{i=1}^{k}$, then in the next comparison between $A[k]$ and $A[k+1]$, (and swap), $A[k+1]$ is the largest one among $\{A[i]\}_{i=1}^{k+1}$. We finish the proof. □

Proof for bubble-sort

Statement

b) if there is no pair of consecutive entries out of order, then the entire array is sorted.

Proof for bubble-sort

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b) if there is no pair of consecutive entries out of order, then the entire array is sorted.

Proof.

1) after the i-th iteration in the outer loop, according to a), we know that

$$A[n], A[n-1], \cdots, A[n-i+1] \text{ are sorted.}$$

Proof for bubble-sort

Statement

b) if there is no pair of consecutive entries out of order, then the entire array is sorted.

Proof.

1) after the i-th iteration in the outer loop, according to a), we know that

$$A[n], A[n-1], \cdots, A[n-i+1] \text{ are sorted.}$$

2) according to the condition, after the i-th iteration, no swap for $A[j]$ with $j = 1, 2, \cdots, n-i$, i.e.,

$$A[1], A[2], \cdots, A[n-i], A[n-i+1] \text{ are sorted.}$$

Accordingly, we finish the proof. □

Remark: Our example finished after the 6th round.

Running time analysis for Bubble-sort

Algorithm 2: Bubble-sort

```
Input: An array A[1, 2, ..., n]
Output: An sorted array A[1, 2, ..., n]
for i = 1, ..., n − 1 do
    for j = 1, ..., n − i do
        if A[j] > A[j + 1] then
            X ← A[j];
            A[j] ← A[j + 1];
            A[j + 1] ← X;
        end
    end
end
```

Runtime analysis results

Runtime analysis: How many comparisons are done.

$$\Rightarrow \text{Total runtime} = \sum_{i=1}^{n-1} \Theta(n-i) = \Theta(n^2)\,.$$

□

Theorem

Bubble-sort algorithm correctly sorts the input array and runs in $\Theta(n^2)$ *time.*

*More information

- the best case time complexity: $\Theta(n)$ if the array is already sorted
- the average case[1] time complexity: $\Theta(n^2)$
  - all possible $n!$ inputs
- space complexity: $\Theta(1)$

[1] check https://cs.stackexchange.com/questions/20/evaluating-the-average-time-complexity-of-a-given-bubblesort-algorithm if you're interested in.