# Discrete Mathematics and Its Applications 2 (CS147)

***Lecture 8: Quick-sort, probability space***

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- randomized algorithms: correct with high probability
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### Problem (Open question in complexity theory)

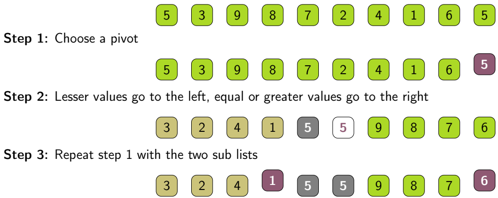
*One of the biggest* **open questions** *in complexity theory is*

*whether randomness really helps?*

- randomized algorithms
- de-randomized techniques

# Quick-sort

- consists of 3 steps:
  - Select a **pivot** from the array
  - partitioning the other elements into two sub-arrays, according to whether they are less than or greater than the pivot.
  - recursively do this: a divide-and-conquer algorithm



# Illustration

5 3 9 8 7 2 4 1 6 5

**Step 1:** Choose a pivot

5 3 9 8 7 2 4 1 6 5

**Step 2:** Lesser values go to the left, equal or greater values go to the right

3 2 4 1 5 5 9 8 7 6

**Step 3:** Repeat step 1 with the two sub lists

3 2 4 1 5 5 9 8 7 6

**Step 4:** Repeat step 2 with the sub lists:

1 3 2 4 5 5 6 9 8 7

**Step 5:** and again and again!

1 3 2 4 5 5 6 9 8 7

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# Deterministic Quick-sort algorithm

**Algorithm 1:** Deterministic Quick-sort

**Input:** An array $A[1, 2, \ldots, n]$
**Output:** An sorted array $A[1, 2, \ldots, n]$

pivot $\leftarrow A[n]$ % we can choose any position we want.;
$S_{\text{smaller}} \leftarrow [], S_{\text{larger}} \leftarrow []$;
**for** $i = 1, \ldots, n$ **do**
  **if** $A[i] \leq$ *pivot* **then**
    $S_{\text{smaller.append}}(A[i])$;
  **end**
  **else** $S_{\text{larger.append}}(A[i])$;
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**return** [Quick-sort($S_{\text{smaller}}$, pivot, $S_{\text{larger}}$) ];

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$$T(n) = T(a) + T(n-a) + \Theta(n)$$

# Running time analysis

**Statement (Worst case $\Theta(n^2)$)**

*If the array is* $\{n, n-1, n-2, \cdots, 2, 1\}$, *a sorted array, then there will be a total of* $\frac{n(n-1)}{2} = \Theta(n^2)$ *comparisons.*

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How to improve it?

- How to choose pivot is important!
- Randomly choose it.

## Statement

*Worst-case expected-time bound is* $\Theta(n \log n)$.

We will prove later in this module.

# Recall some knowledge about set theory...

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- The **power set** of $A$ is the collection of all of its subsets, i.e., $2^A = \{B : B \subset A\}$.

# Randomness and sample space

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## Example

- toss a coin twice: (head, head), (head, tail), (tail,head), (tail,tail)
- $\Omega = \{HH, HT, TH, TT\}$

# Events

## Definition

We define *event* as a set of outcomes, denoted as $A \subseteq \Omega$. We call an *event* occurs if and only if some sample point(s) included in $A$ occur.

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## Example (Experiment with countably infinite outcomes)

Consider an experiment: keep tossing a coin until the head appears.

- countably infinite outcomes: H, TH, TTH, TTTH, $\cdots$
- we're interested in:
  $A_k =$ "H appears exactly in the $k$-th toss"

# Property of events

## Property (using Venn diagram)

*An event is a set!*

- *complement:* $A^c = \{\omega : \omega \notin A\}$
- *union:* $A \cup B = \{\omega : \omega \in A, \text{ or } \omega \in B\}$
- *intersection:* $A \cap B = \{\omega : \omega \in A, \text{ and } \omega \in B\}$
- *difference:* $A - B = \{\omega : \omega \in A, \text{ and } \omega \notin B\}$
- *symmetric difference:* $A \Delta B = (A - B) \cup (B - A)$
- *Event $A$ and $B$ are called disjoint if $A \cap B = \emptyset$.*

**Remark:** Not arbitrary event can be assigned to a probability.

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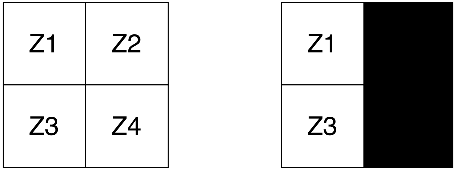
◦ $(\Omega, \mathcal{F})$ is called a measurable space

# *Example: Intuition on a set of “measurable" events

○ a small ball in a box equally split into four regions (from the front): Z1, Z2, Z3, Z4
○ shake the box and the ball rolls randomly
○ which region does the ball stay?

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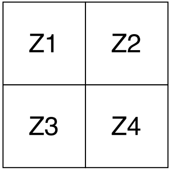


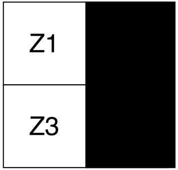


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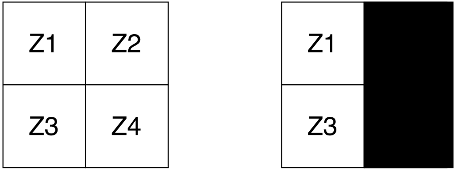


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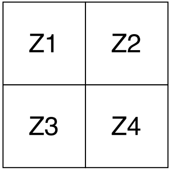


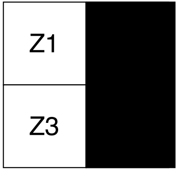


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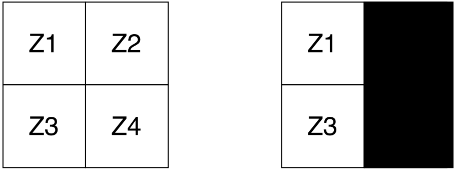


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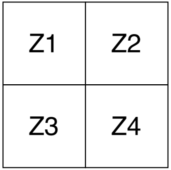


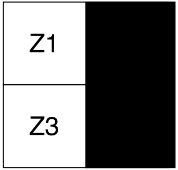


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the event $\{Z2\}$ is $\mathcal{F}_1$-*measurable* but not $\mathcal{F}_2$-*measurable*!

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## Definition (Probability)

Probability $\Pr : \mathcal{F} \to [0, 1]$ is a function that assigns a value to events

- nonnegativity: $\Pr(A) \geq 0$
- normalization: $\Pr(\Omega) = 1$
- countable additivity: if $A_i \in \mathcal{F}$ is a countable sequence of disjoint sets, then $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$

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- $(\Omega, \mathcal{F})$ is a measurable space
- $(\Omega, \mathcal{F}, \Pr)$ is a probability space

## Properties of probability

- $\forall A \in \mathcal{F}$, we have $\Pr(A^c) = 1 - \Pr(A)$.
- If $A, B \in \mathcal{F}$ and $A \subseteq B$, then $\Pr(B) = \Pr(A) + \Pr(B - A) \geq \Pr(A)$.
- If $A, B \in \mathcal{F}$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.