Discrete Mathematics and Its Applications 2 (CS147)

Lecture 7: Recurrence relations and generating function

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Target: solving recurrence equation

Definition (Recurrence equation or difference equation)

$$a_n = f(a_{n-1}, a_{n-2}, \cdots, a_0)\,, \quad \text{under certain initializations.}$$

Remark: a) f is a given function
b) depending on some or all of its past values $a_{n-1}, a_{n-2}, \cdots$.

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$a_n = 4a_{n-1} - 2$ for $n \geq 2$ and initialization $a_1 = 2$.

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Example (Fibonacci sequence)

$a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ and initialization $a_1 = 0, a_2 = 1$.

Definitions in LINEAR recurrence equations

Definition (linear)

A recurrence equation is called **linear** if it can be rewritten a linear function of its past values $a_{n-1}, a_{n-2}, \cdots$.

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Definition (homogeneous)

A linear recurrence equation is called **homogeneous** if a_n only depends on its past values $a_{n-1}, a_{n-2}, \cdots$.

Generating function

Definition

The generating function for the sequence $a_0, a_1, \ldots, a_n, \ldots$ of real number is given by

$$G(x) := a_0 + a_1 x + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n \,.$$

Solving Nonhomogeneous, Constant Coefficients, and Linear Difference Equations...

Examples: from sequence to $G(x)$

Example

The generating function for the sequence $1, 1, \ldots$ is

$$G(x) := 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1\,.$$

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Example

The generating function for the sequence $1, a, a^2, a^3, \ldots$ is

$$G(x) := 1 + ax + a^2x^2 + \ldots = \sum_{n=0}^{\infty} a^n x^n = \frac{1}{1-ax} \quad \text{for } |ax| < 1\,.$$

Examples: from $G(x)$ to sequences

Example

Let $G(x) = \frac{1}{(1-x)^2}$, find the coefficients $a_0, a_1, \dots$ in the expansion $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

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Proof.

Recall $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}(x^n) = 0 + \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}(x^n).$$

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$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n,$$

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n.$$

□

Some useful power series and their closed forms

[remember to check the convergence condition in the calculus textbook.]

$$\frac{1}{1-ax} = \sum_{i=0}^{\infty} a^i x^i$$

$$\frac{1}{(1-ax)^2} = \sum_{i=0}^{\infty} (i+1) a^i x^i$$

$$\frac{1}{(1+ax)^n} = \sum_{i=0}^{\infty} \binom{-n}{i} a^i x^i$$

$$\ln(1+ax) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} a^i x^i$$

$$\exp(ax) = \sum_{i=0}^{\infty} \frac{1}{i!} a^i x^i$$

Using generating function to solve recurrence functions

Example

Considering the following iteration:

$$a_n = 8a_{n-1} + 10^{n-1}\,, \forall n \geq 1\,, \text{with } a_0 = 1\,.$$

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Way 1.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} (8a_{n-1} + 10^{n-1}) x^n$$

$$= 1 + 8\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n = 1 + 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \frac{x}{1-10x}$$

$$= 1 + 8x \sum_{n=0}^{\infty} a_n x^n + \frac{x}{1-10x} = 1 + 8xG(x) + \frac{x}{1-10x}\,.$$

□

Way 2.

Recall $a_n = 8a_{n-1} + 10^{n-1}$ with $n \geq 1$,

$$
\begin{array}{rl}
G(x) = & a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \\
8xG(x) = & \quad +8a_0 x + 8a_1 x^2 + \cdots + 8a_{n-1} x^n + \cdots \\
\hline
(1-8x)G(x) = & a_0 + 10^0 x + 10^1 x^2 + \cdots + 10^{n-1} x^n + \cdots
\end{array}
$$

Way 2.

Recall $a_n = 8a_{n-1} + 10^{n-1}$ with $n \geq 1$,

$$\begin{array}{rl} G(x) = & a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \\ 8xG(x) = & \quad +8a_0x + 8a_1x^2 + \cdots + 8a_{n-1}x^n + \cdots \\ \hline (1-8x)G(x) = & a_0 + 10^0x + 10^1x^2 + \cdots + 10^{n-1}x^n + \cdots \end{array}$$

$$\Rightarrow (1-8x)G(x) = 1 + \frac{10^0 x}{1-10x}$$

□

Solutions

To be continued.

$$G(x)(1-8x) = 1 + \frac{x}{1-10x} = \frac{1-9x}{1-10x},$$

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which implies

$$\begin{aligned} G(x) &= \frac{1-9x}{(1-10x)(1-8x)} \\ &\triangleq \frac{A}{1-10x} + \frac{B}{1-8x}. \end{aligned}$$

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That means

$$(A+B) - (8A+10B)x = 1-9x\,,$$

that means $A+B=1$ and $8A+10B=9$. We have $A=B=\frac{1}{2}$. □

Solutions

To be continued.

Accordingly, we have

$$\begin{aligned} G(x) &= \frac{1}{2}\frac{1}{1-10x} + \frac{1}{2}\frac{1}{1-8x} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(10x)^n + \frac{1}{2}\sum_{n=0}^{\infty}(8x)^n\,, \end{aligned}$$

Solutions

To be continued.

Accordingly, we have

$$G(x) = \frac{1}{2}\frac{1}{1-10x} + \frac{1}{2}\frac{1}{1-8x}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}(10x)^n + \frac{1}{2}\sum_{n=0}^{\infty}(8x)^n ,$$

which implies

$$G(x) = \sum_{n=0}^{\infty}\left(\frac{1}{2}(10^n + 8^n)\right)x^n .$$

$$\Rightarrow a_n = \frac{1}{2}(10^n + 8^n), \quad \forall n \geq 1 .$$

□

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- obtain the formulation of $G(x)$
  - taking a_n into $G(x)$
  - remove the recurrence from $G(x)$ and simplify it
  - partial fraction decomposition of a rational expression
- series expansion and summation
- equate the coefficients of x^n

* Characteristic root method[1]

◦ normally, the order is smaller than 2

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad \text{with initializations on} a_1, a_2 .$$

[1] https://itk.ilstu.edu/faculty/chungli/DIS300/dis300chapter8.pdf if you're interested in.

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- find two roots r_1 and r_2
  ○ the roots are different, $a_n = A_1 r_1^n + A_2 r_2^n$
  ○ the roots are the same, $a_n = (A_1 + A_2 n) r_1^n$
- two unknown constants A_1 and A_2 are determined by the given initial conditions

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Non-homogeneous part: a bit complex...

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